RBC MODELS

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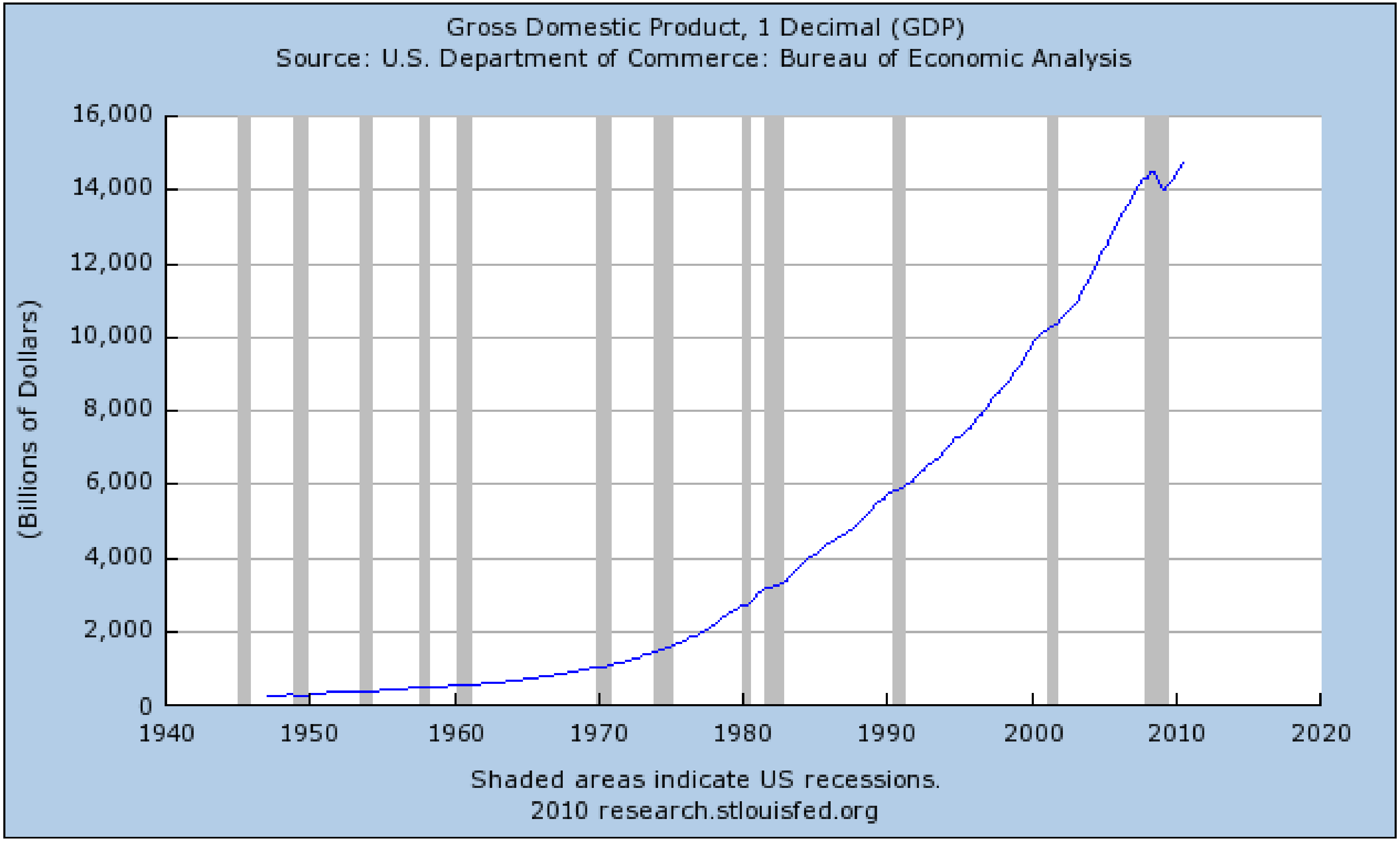
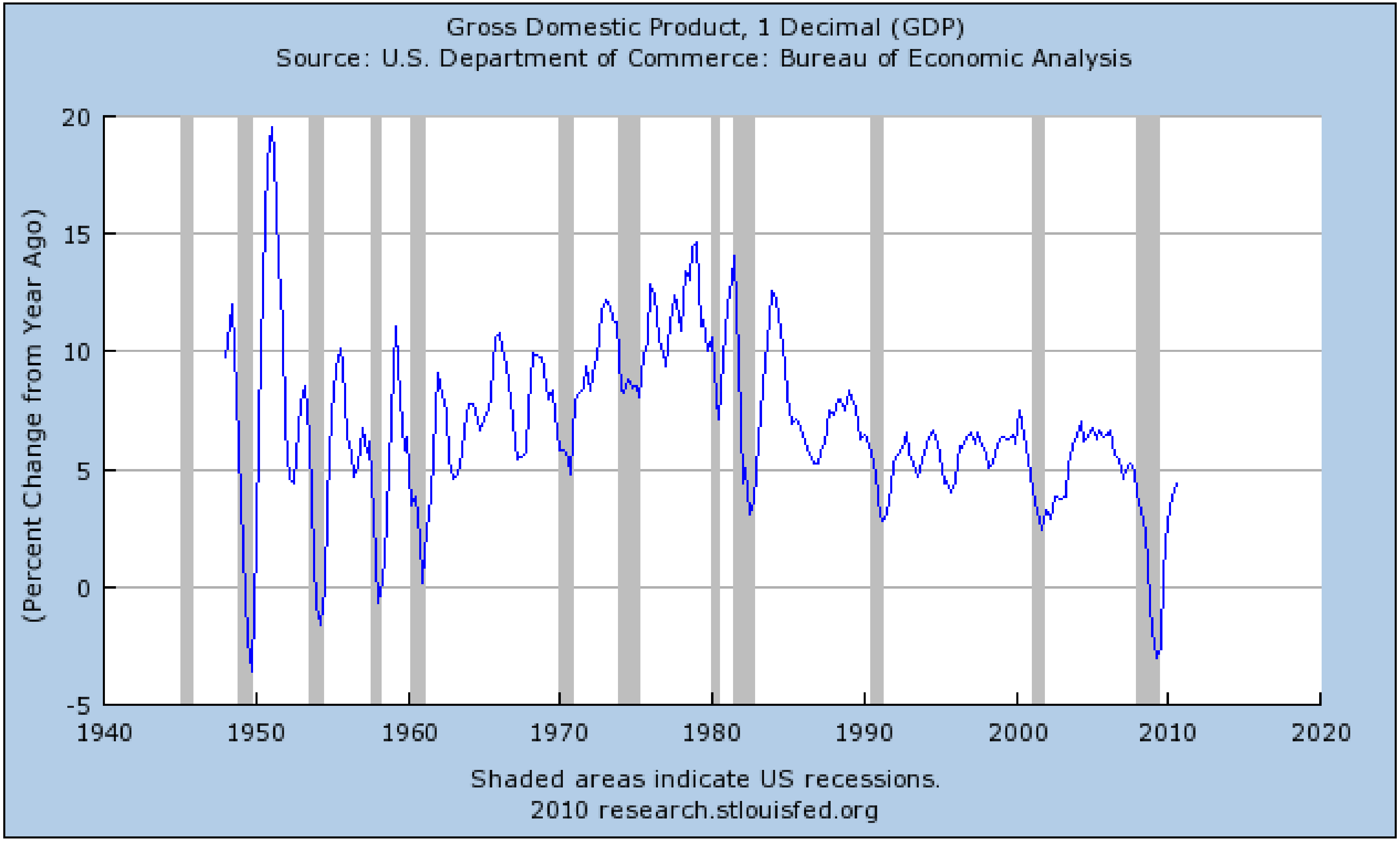
Recessions in the United States since World War II

Year and quarter of peak in real GDP	Number of quarters until trough in real GDP	% Change in real GDP, Peak to trough
1948:4	2	-1.8
1953:2	3	-2.7
1957:3	2	-3.7
1960:1	3	-1.6
1970:3	1	-1.1
1973:4	5	-3.1
1980:1	2	-2.2
1981:3	2	-2.9
1990:3	2	-1.3
2001:2	1	-0.4
2007:4	6	

U.S Real GDP


Gross Domestic Product, 1 Decimal (GDP)
Source: U.S. Department of Commerce: Bureau of Economic Analysis
16,000
14,000
12,000
10,000
8,000
6,000
4,000
2,000
0
(Billions of Dollars)
1940
1950
1960
1970
1980
1990
2000
2010
2020
Shaded areas indicate US recessions.
2010 research.stlouisfed.org

Growth Rate of U.S. Real GDP



Behavior of the components of output in recessions

Component of GDP	Average Share in GDP (%)	Average share in fall in GDP in recessions relative to normal growth (%)
Consumption		
Durables	8.5	15.1
Nondurables	25.4	10.3
Services	30.4	9.5
Investment		
Residential	4.8	10.7
Fixed nonresidential	10.6	20.3
Inventories	0.6	41.8
Net Exports	-0.6	-11.4
Government purchases	20.3	3.8

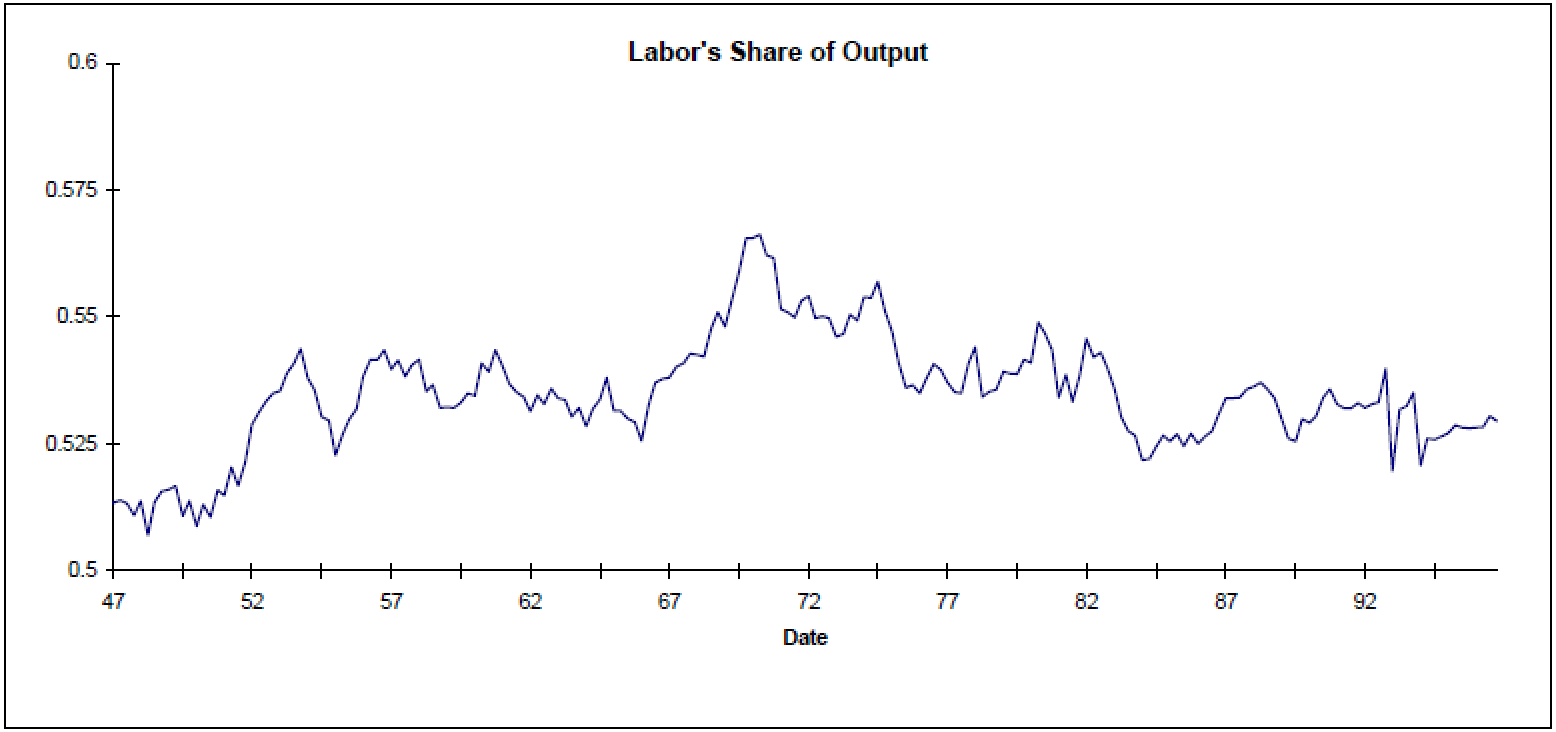
Behavior of some important macroeconomic variables in recessions

Variable	Average change in recessions	Number of recessions in which variable falls
Real GDP	-3.9	10/10
Employment	-2.8	10/10
Unemployment rate	+1.6	0/10
Hours	-2.2	10/10
Output per hour	-1.7	9/10
Inflation	-0.1	4/10
Real compensation per hour	-0.6	7/10
Nominal interest rate 3-m T	-1.5	9/10
Ex post real interest rate	-1.2	7/10
Real money stock (M2/GDP deflator)	-0.9	3/7

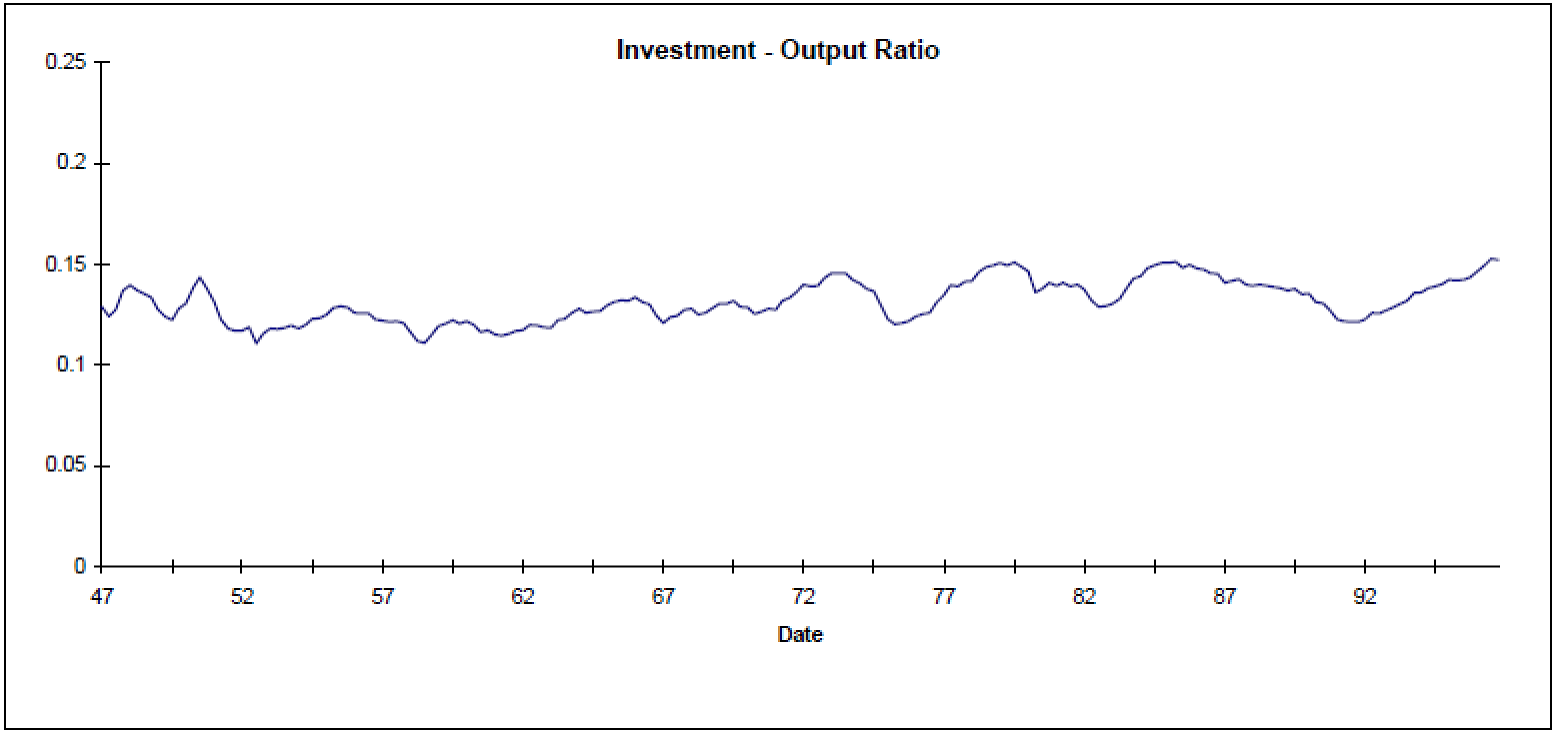
Business Cyclical Statistics for the U.S. Economy

Variable	Volatility (% Standard Deviation)	Relative Volatility	First Order Auto-correlation	Contemporaneous Correlation with Output
Y: Output	1.81	1	0.84	
C: Consumption	1.35	0.74	0.80	0.88
I: Investment	5.30	2.93	0.87	0.80
K: Capital Stock	0.63	0.36		0.04
N: Employment	1.79	0.99	0.88	0.88
Y/N	1.02	0.56	0.74	0.55
w	0.68	0.38	0.66	0.12
r	0.30	0.16	0.60	-0.35
A	0.98	0.54	0.74	0.78

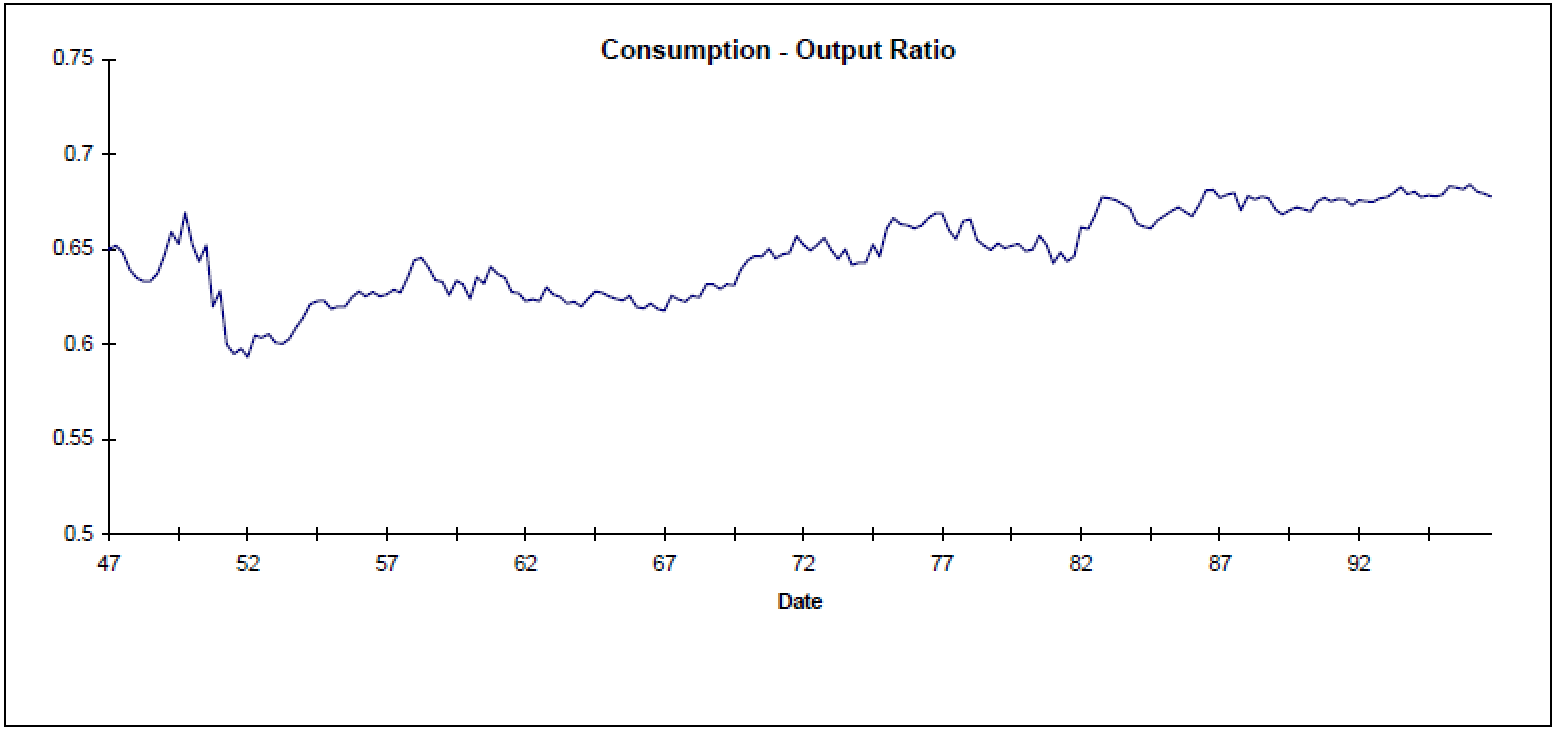
Great ratios: labor share



Great ratios: I/Y



Great ratios: C/Y



Household's Program

Household's chooses a sequence of consumption C_t, labor N_t, investment X_t, and next period capital K_{t+1} to

$$\max \; E_0 \sum_{t=0}^{\infty} \beta^t \left[\tfrac{1}{1-\gamma} C_t^{1-\gamma} - \tfrac{1}{1+\gamma_n} N_t^{1+\gamma_n} \right]$$

subject to the sequence of budget constraints

$$C_t + X_t + \phi(X_t, K_t) - T_t = r_t^k K_t + W_t N_t$$

and the capital accumulation process

$$K_{t+1} = X_t + (1-\delta) K_t$$

where $\phi(\cdot,\cdot)$ are increasing and convex investment adjustment costs, r_t^k is the rental rate of capital, δ is its depreciation rate, W_t is the wage rate, π_t are profits from the ownership of firms, and T_t are lump-sum taxes.

Household's Program

Form the Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t \left[\tfrac{1}{1-\gamma} C_t^{1-\gamma} - \tfrac{1}{1+\gamma_n} N_t^{1+\gamma_n} \right] + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[r_t^k K_t + W_t N_t - C_t - X_t - \phi(X_t, K_t) \right]$$

$$+ \sum_{t=0}^{\infty} \beta^t \mu_t \left[X_t + (1-\delta) K_t - K_{t+1} \right]$$

Optimality conditions include:

marginal utility of wealth: $C_t^{-\gamma} = \lambda_t$

labor supply: $N_t^{\gamma_n} = \lambda_t W_t$

investment demand: $\frac{\mu_t}{\lambda_t} \equiv q_t = 1 + \frac{\partial \phi(X_t, K_t)}{\partial X_t}$

capital supply: $q_t = \beta \frac{\lambda_{t+1}}{\lambda_t} \left[r_{t+1}^k - \frac{\partial \phi(X_{t+1}, K_{t+1})}{\partial K_{t+1}} + q_{t+1}(1-\delta) \right]$

Household's Program

Form the Lagrangian

$$\mathrm{L} = \sum_{t=0}^{\infty} \beta^t \left[\tfrac{1}{1-\gamma} C_t^{1-\gamma} - \tfrac{1}{1+\gamma_n} N_t^{1+\gamma_n} \right] + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[r_t^k K_t + W_t N_t - C_t - X_t - \phi(X_t, K_t) \right]$$

$$+ \sum_{t=0}^{\infty} \beta^t \mu_t \left[X_t + (1-\delta) K_t - K_{t+1} \right]$$

$$\frac{\partial L}{\partial X_t} = -\beta^t \lambda_t - \beta^t \lambda_t \frac{\partial \phi(X_t, K_t)}{\partial X_t} + \beta^t \mu_t = 0$$

$$\frac{\partial L}{\partial K_{t+1}} = -\beta^t \mu_t + \beta^{t+1} \lambda_{t+1} r_{t+1}^k - \beta^{t+1} \lambda_{t+1} \frac{\partial \phi(X_{t+1}, K_{t+1})}{\partial K_{t+1}} + \beta^{t+1} \mu_{t+1} (1-\delta) = 0$$

$$\frac{\partial L}{\partial X_t} = 0 \quad \Rightarrow \quad \frac{\mu_t}{\lambda_t} \equiv q_t = 1 + \frac{\partial \phi(X_t, K_t)}{\partial X_t}$$

$$\frac{\partial L}{\partial K_{t+1}} = 0 \quad \Rightarrow \quad \frac{\mu_t}{\lambda_t} = \beta \frac{\lambda_{t+1}}{\lambda_t} \left[r_{t+1}^k - \frac{\partial \phi(X_{t+1}, K_{t+1})}{\partial K_{t+1}} + \frac{\mu_{t+1}}{\lambda_{t+1}} (1-\delta) \right]$$

$$q_t = \beta \frac{\lambda_{t+1}}{\lambda_t} \left[r_{t+1}^k - \frac{\partial \phi(X_{t+1}, K_{t+1})}{\partial K_{t+1}} + q_{t+1} (1-\delta) \right]$$

Firms' Program

Each period perfectly competitive firms rent capital K_t and hire labor N_t to maximize profits π_t given by:

$$\pi_t = Y_t - r_t^k K_t - W_t N_t$$

subject to the production function

$$Y_t = K_t^{1-\alpha}\left(A_t N_t\right)^{\alpha}$$

where A_t denotes the level of technology. Optimality conditions include:

Capital demand:

$$r_t^k = (1-\alpha) K_t^{-\alpha}\left(A_t N_t\right)^{\alpha} = (1-\alpha)\frac{Y_t}{K_t}$$

Labor demand:

$$W_t = \alpha K_t^{1-\alpha} A_t^{\alpha} N_t^{\alpha-1} = \alpha \frac{Y_t}{N_t}$$

Government's Program and Aggregate Resource Constraint

Each period the government collects taxes from households T_t to finance government expenditure G_t. Assume that each period the government runs a balanced budget such that

$$T_t = G_t$$

Finally, the aggregate resource constraint is :

$$Y_t = C_t + X_t + \phi\left(X_t, K_t\right) + G_t$$

General Equilibrium

Characterized by a vector of prices $\left(r_t^k, W_t, q_t\right)$ and three vectors of demand and supply levels - $\left(C_t, X_t, N_t, K_{t+1}, T_t\right)$ for the household, $\left(Y_t, N_t, K_t\right)$ for the firm, and $\left(T_t, G_t\right)$ for the government - such that

∘ each agent behaves optimally (their decisions solve the previously described programs);

∘ each market clears:

- $K_t^{1-\alpha}\left(A_t N_t\right)^{\alpha} = C_t + X_t + \phi\left(X_t, K_t\right) + G_t$ (goods market)
- $N_t^{\gamma_n} C_t^{\gamma} = W_t = \alpha \frac{Y_t}{N_t}$ (labor market)
- $\frac{1}{\beta} \frac{C_{t+1}^{\gamma}}{C_t^{\gamma}} q_t + \frac{\partial \phi\left(X_{t+1}, K_{t+1}\right)}{\partial K_{t+1}} - q_{t+1}\left(1-\delta\right) = r_{t+1}^k = \left(1-\alpha\right) \frac{Y_{t+1}}{K_{t+1}}$ (capital market)

Alternative Specification: Household's Program

The consumer will choose a sequence of consumption C_t, labor N_t, and firms' shares S_{t+1} to

$$\max \; E_0 \sum_{t=0}^{\infty} \beta^t u(C_t, N_t)$$

subject to

$$Q_t S_{t+1} = (D_t + Q_t) S_t + W_t N_t - C_t - T_t$$

where Q_t is the price of the shares that pay dividends D_t, while W_t is the wage rate and T_t are lump-sum taxes.

Alternative Specification: Household's Optimality Conditions

Using λ_{t} to denote the Lagrangian multiplier on the constraint, each period t, the household's optimality conditions include:

Marginal utility of wealth:

$$\frac{\partial u(C_t, N_t)}{\partial C_t} = \lambda_{\mathrm{t}}$$

Labor supply:

$$-\frac{\partial u(C_t, N_t)}{\partial N_t} = \lambda_{\mathrm{t}} W_t$$

Equity demand (savings supply):

$$Q_t \lambda_{\mathrm{t}} = \beta E_t \left\{ \lambda_{\mathrm{t+1}} \left(Q_{t+1} + D_{t+1} \right) \right\}$$

Alternative Specification: Firms' Program

Period perfectly competitive firms choose a sequence of investment X_t, capital K_{t+1} and labor N_t to maximize the value for shareholders given by:

$$E_0\left\{\sum_{t=0}^{\infty}\frac{D_t}{R^t}\right\}$$

subject to

$$D_t = Y_t - W_t N_t - X_t - \phi\left(X_t, K_t\right)$$

$$Y_t = K_t^{1-\alpha}\left(A_t N_t\right)^{\alpha}$$

$$K_{t+1} = X_t + \left(1-\delta\right)K_t$$

where R is the riskless gross rate of return on riskless bonds, A_t denotes the level of technology, $\phi(\cdot,\cdot)$ are investment adjustment costs and δ is the capital depreciation rate.

Alternative Specification: Firms' Program Optimality Conditions

Form the Lagrangian :

$$\text{L} = \sum_{t=0}^{\infty} \frac{1}{R^t} \left\langle \begin{array}{l} D_t + \lambda_t^D \left[Y_t - W_t N_t - X_t - \phi(X_t, K_t) - D_t \right] \\ + \lambda_t^Y \left[K_t^{1-\alpha} (A_t N_t)^{\alpha} - Y_t \right] + \lambda_t^K \left[X_t + (1-\delta) K_t - K_{t+1} \right] \end{array} \right\rangle$$

The optimality conditions include :

$$\frac{\partial L}{\partial D_t} = 0 \quad \Rightarrow \quad 1 - \lambda_t^D = 0$$

$$\frac{\partial L}{\partial Y_t} = 0 \quad \Rightarrow \quad \lambda_t^D - \lambda_t^Y = 0$$

$$\frac{\partial L}{\partial N_t} = 0 \quad \Rightarrow \quad -\lambda_t^D W_t + \lambda_t^Y \alpha \frac{Y_t}{N_t} = 0$$

$$\frac{\partial L}{\partial X_t} = 0 \quad \Rightarrow \quad -\lambda_t^D \left(1 + \frac{\partial \phi(X_t, K_t)}{\partial X_t} \right) + \lambda_t^K = 0$$

$$\frac{\partial L}{\partial K_{t+1}} = 0 \quad \Rightarrow \quad -\lambda_t^K + \frac{1}{R} \left(-\lambda_{t+1}^D \frac{\partial \phi(X_{t+1}, K_{t+1})}{\partial X_t} + \lambda_{t+1}^Y (1-\alpha) \frac{Y_{t+1}}{K_{t+1}} + \lambda_{t+1}^K (1-\delta) \right) = 0$$

Alternative Specification: Firms' Program Optimality Conditions

We can simplify these optimality conditions using $\lambda_t^D = \lambda_t^Y = 1$.

Also notice that to finance capital accumulation firms need to issue shares, therefore optimality requires equalizing the marginal benefit and cost $\lambda_t^K \equiv Q_t$.

The optimal labor demand, investment and equity supply conditions are:

$$\frac{\partial L}{\partial N_t} = 0 \quad \Rightarrow \quad W_t = \alpha \frac{Y_t}{N_t}$$

$$\frac{\partial L}{\partial X_t} = 0 \quad \Rightarrow \quad Q_t = 1 + \frac{\partial \phi(X_t, K_t)}{\partial X_t}$$

$$\frac{\partial L}{\partial K_{t+1}} = 0 \quad \Rightarrow \quad Q_t = \frac{1}{R}\left((1-\alpha)\frac{Y_{t+1}}{K_{t+1}} - \frac{\partial \phi(X_{t+1}, K_{t+1})}{\partial X_t} + Q_{t+1}(1-\delta) \right)$$

Notice the similarity between these optimality conditions and those that we got in the model where households invest. To complete the model we just need to describe the government and the aggregate constraint.

Steady-State

In steady-state $K_{t+1} = K_t = K$ then the capital accumulation equation implies

$$K = X + (1-\delta)K \qquad \Rightarrow \qquad X = \delta K$$

In general the properties of the investment adjustment cost function are such that $\phi(\delta K, K) = 0$, therefore the price of shares (capital) in steady-state is $Q = 1$. Using $Q = 1$ in the equity demand condition we can derive the steady-state output-capital ratio as

$$1 = \tfrac{1}{R}\left((1-\alpha)\tfrac{Y}{K} + (1-\delta)\right) \qquad \Rightarrow \qquad \frac{Y}{K} = \frac{R-(1-\delta)}{(1-\alpha)}$$

We get a constant $\frac{\mathrm{Y}}{\mathrm{K}}$ similar to what we observe in the data. Also note that

$$\frac{\mathrm{X}}{\mathrm{Y}} = \frac{\frac{X}{K}}{\frac{Y}{K}} = \frac{\delta}{\frac{R-(1-\delta)}{(1-\alpha)}}$$

Calibration

Usually we will pick parameter values consistent with microeconomic evidence. We also require the parameter values to be consistent with empirical evidence of the great ratios.

For example King and Rebelo notice that the average return to capital from the Standard and Poor 500 index over the 1948-1986 period is 6.5% per annum, therefore in our quarterly model $R = \frac{6.5\%}{4}$.

We have discussed that the share of capital return on output $\frac{\text{RK}}{\text{Y}} = (1-\alpha) = \frac{1}{3}$ then $\alpha = \frac{2}{3}$.

Calibration

So the only parameter missing to pin down the steady-state $\frac{X}{Y}$ is the quarterly depreciation rate $\frac{\delta}{4}$.

In the U.S. data the average $\frac{X}{Y} \approx 20\%$. Then we set the quarterly depreciation rate $\delta = 2.5\%$.